Math 221: Final Exam Extras and Theory Review Sheet Answers

Ratio/Proportion Problems:

1. The ratio of girls to boys us 27:36 = 3:4. There are 63 students in the class, so the girls to class ratio is 27:63 = 3:7 and the boys to class ratio is 36:63 = 4:7.
2. Draw a bowl of mashed potatoes with 1 cup gravy and 8 cups potatoes. Since we only need 2 cups of mashed potatoes, cut this picture into fourths, which leaves 1/4 cups of gravy.
3. The total product if using 1 cup rice and 2 cups water is 3 cups, so the total product to uncooked rice ratio is 3:1. Draw a picture of 2 cups of water (draw two actual figures of 1 cup for this) and 1 cup of rice. Since we want 4 cups of uncooked rice, duplicate this figure so that you have 4 of them. You then have 8 cups of water and 12 cups of total product (counting everything).
4. $\frac{1300}{1000000} = \frac{x}{1800000}$. Cross multiplying, we get $2340000000 = 1000000x$, so $x = 2340$ pollen particles.
5. We need the ratio of teenagers to total people. For every 3 teenagers, there are 2 adults, so 5 people total, giving a teenagers to total people ratio of 3:5. Now, $\frac{3}{5} = \frac{x}{180}$. Cross multiplying, we get $540 = 5x$, so $x = 108$.

Converting Fractions to Percents

1. $\frac{21}{25} \times \frac{4}{4} = \frac{84}{100} = 84\%$.
2. $\frac{21}{30} = \frac{7}{10} \cdot \frac{10}{10} = \frac{70}{100} = 70\%$.
3. $\frac{17}{20} \times \frac{5}{5} = \frac{85}{100} = 85\%$.

Converting Percents to Fractions

1. $64\% = \frac{64}{100} = \frac{16}{25}$.
2. $15.5\% = \frac{15.5}{100} \cdot \frac{10}{10} = \frac{155}{100} = \frac{31}{20}$.
3. $16.\overline{6}\% = 0.1\overline{6} = \frac{1}{6}$.

Percent Problems

1. We are computing 25% of 188, which is $188 \cdot .25 = 47$. So, Marc sold 47 boxes of candy.
2. Let n be the value of the original paycheck. Then $n - 20\%n = 720$ since 20% or the original paycheck was deducted. So, $n - 0.2n = 720 \Rightarrow 0.8n = 720 \Rightarrow n = 900$. So, Michael's original paycheck was \$900.
3. Let n be the percent discount as a decimal. Then $45 - 45n = 27$ since the original price times the percent discount is deducted from the original price. So, $45 = 45n + 27 \Rightarrow 45n = 18 \Rightarrow n = \frac{18}{45} = \frac{2}{5} = 40\%$. Thus, the dress was discounted at 40% off.
4. Let n be the percent increase as a decimal. Then $165000 + 165000n = 198000$ since the original price times the percent increase is added to the original price. So, $165000n = 33000 \Rightarrow n = \frac{33000}{165000} = \frac{1}{5} = 20\%$. Thus, Mort's house increased in value by 20%.
5. Let n be the original wages from last year. Then $n + 0.05n = 19929$ since the original wages times the percent increase is added to the original wages to get the new wages. So, $1.05n = 19929 \Rightarrow n = 18980$. Thus, Gail's wages were \$18980.

Test 2 Material Explanations

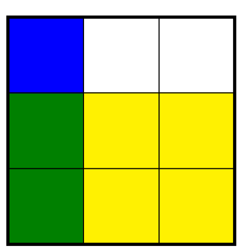
1. • (Addition) Consider $29 + 13$. When we add $9 + 2$, we get 12 units, but we need to convert 10 of those units to a long. This gives us 2 below and we write a 1 above the tens column to denote the new long.

- (Subtraction) Consider $22 - 9$. When we try to subtract 9 from 2, we don't have enough units. So, we take one of our 2 longs and convert it to 10 units, giving us 1 long and 12 units. We can then subtract 9 from 12 as normal.
- (Multiplication) Consider 13×14 and use the array model to show the multiplication. When we multiply 13 times 4, that gives us 12 units, of which we change 10 of them to a long, and then 4 more longs, giving 5 total. We then multiply 13 times 1, but this is actually 13 times 10. In our array model, we are crossing a long with a long and 3 units, so we get 1 flat and 3 longs. So, we then add these and combine them as necessary.

2. Consider $22 - 9$. When we add 1 to each to make $23 - 10$, we are actually doing the following: $(22+1) - (9+1)$. Notice that if we distribute our negative sign, we get $22 + 1 - 9 - 1 = 22 - 9$. Essentially, we are adding some value and then subtracting it right back out, so the problem is equivalent but easier to solve.
3. - (Addition) Consider $29 + 13$. When we add $9 + 3$, we get 12, and we put the 1 in the upper triangle and 2 in the lower triangle. However, when we then add up the lattices, the 1 is in the diagonal for the tens place, where it should be.
   - (Multiplication) This is too difficult to do without pictures, so see the quiz 5 solutions.
4. Consider $126 \div 5$. We have 5 groups that we are trying to split everything into, so we first look at our 1 flat. It won't go into the 5 groups, so we convert it to longs, giving us 12 longs. Now, we can put 2 longs in each group and use up 10, leaving 2 longs behind. Now, these won't go into the 5 groups, so we convert them to units, giving us 16 units. Now, we can put 3 units in each group and use up 15, leaving 1 unit behind. This unit is our remainder.
5. (2) Since 10 is divisible by 2, any multiple of 10 is divisible by 2. So, for any number, if the last digit is divisible by 2, the rest of the number is since it is a multiple of 10 plus the last digit.
   (4) Since 100 is divisible by 4, any multiple of 100 is divisible by 4. So, for any number, if the last two digits are divisible by 4, the rest of the number is since it is a multiple of 100 plus the last two digits.
   (5) Since 10 is divisible by 5, any multiple of 10 is divisible by 5. So, for any number, if the last digit is divisible by 5, the rest of the number is since it is a multiple of 10 plus the last digit.
   (6) Any number that is divisible by 2 and 3 is also divisible by their least common multiple, 6.
   (8) Since 1000 is divisible by 8, any multiple of 1000 is divisible by 8. So, for any number, if the last three digits are divisible by 8, the rest of the number is since it is a multiple of 1000 plus the last 3 digits.
   (10) If a number ends in a zero, then division by 10 removes that zero.

<u>Test 3 Material Explanations</u>

1. Consider $\frac{1}{3} \times \frac{2}{3}$. Then in the figure below, the intersection of the shaded regions is the numerator, which has 1×2 squares, and the total number of squares is the denominator with $3 \times 3 = 9$ squares.



2. Consider $1.520 = 1.52$. $1.520 = 1 + \frac{5}{10} + \frac{2}{100} + \frac{0}{1000}$, but the last term is just adding zero. So, we have that this $= 1 + \frac{5}{10} + \frac{2}{100} = 1.52$.

3. Consider 1.2×1.1. When we multiply, we get 132, then we move the decimal place 2 places to the left, giving us 1.32, because the two numbers had one decimal place each. This is because $1.2 \times 1.1 = \frac{12}{10} \times \frac{11}{10} = \frac{12\times 11}{100} = \frac{132}{100} = 1.32$.
4. Consider $10 \div 4 = 2.5$. When we divide into the 10, we get a remainder of 2. But 4 can't divide evenly into 2 units, so we convert those two units into 20 tenths. This is the same as adding the decimal place and 0, then bringing down the 0. So, now we would take our 20 tenths and put 5 in each group, giving us our .5 in our answer.
5. Consider $9 \div 4 = 2.25$. When we divide into the 9, we get a remainder of 1. We then add the .0 by the previous problem and we get a remainder of 2. Now, when we add another zero, we are converting these 2 tenths to 20 hundredths, and then we put 5 in each group, giving us the 5 in our answer.
6. Consider $42.31 \div 1.7$. We can write the following: $\frac{42.31}{1.7} \times \frac{10}{10} = \frac{423.1}{17}$. So, we are essentially just multiplying and dividing by 10, which doesn't change the problem.

Proofs

In problems 1-4, we use the common denominator bd. The step where we multiply by b or d in the numerator and denominator is justified by the Fundamental Law of Fractions. The step where we write $\dfrac{c \cdot b}{d \cdot b} = \dfrac{bc}{bd}$ is the commutative property. The last step for is the appropriate definition.

1. The following steps can be performed forwards or backwards: $\dfrac{a}{b} = \dfrac{c}{d}$, $\dfrac{a \cdot d}{b \cdot d} = \dfrac{c \cdot b}{d \cdot b}$, $\dfrac{ab}{bd} = \dfrac{bc}{bd}$, $ac = bd$. ■
2. The following steps can be performed forwards or backwards: $\dfrac{a}{b} > \dfrac{c}{d}$, $\dfrac{a \cdot d}{b \cdot d} > \dfrac{c \cdot b}{d \cdot b}$, $\dfrac{ab}{bd} > \dfrac{bc}{bd}$, $ac > bd$. ■
3. The following steps can be performed forwards or backwards: $\dfrac{a}{b} = \dfrac{c}{d}$, $\dfrac{a \cdot d}{b \cdot d} < \dfrac{c \cdot b}{d \cdot b}$, $\dfrac{ab}{bd} < \dfrac{bc}{bd}$, $ac < bd$. ■
4. $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{a \cdot d}{b \cdot d} + \dfrac{c \cdot b}{d \cdot b} = \dfrac{ab}{bd} + \dfrac{bc}{bd} = \dfrac{ab + bc}{bd}$. ■
5. $\dfrac{a}{b} - \dfrac{c}{d} = \dfrac{a}{b} + \dfrac{^-c}{d} = \dfrac{ad + b\,(^-c)}{bd}$ (by previous proof) $= \dfrac{ad - bc}{bd}$. ■
6. $\dfrac{a}{b} \div cd = \dfrac{\frac{a}{b}}{\frac{c}{d}} = \dfrac{\frac{a}{b} \cdot \frac{d}{c}}{\frac{c}{d} \cdot \frac{d}{c}} = \dfrac{\frac{a}{b} \cdot \frac{d}{c}}{1} = \dfrac{a}{b} \cdot \dfrac{d}{c}$. ■
7. Since we have an if and only if, we need to assume the left side and prove the right (denoted ($\Rightarrow$)) and then assume the right side and prove the left (denoted ($\Leftarrow$)). Note that this proof is simpler than the previous one and is fine to use as is.

   ($\Rightarrow$) Since $\dfrac{a}{b}$ is a terminating decimal, it can be written as a power of 10: $\dfrac{a}{b} = \dfrac{n}{10^k}$, where n is the digits and k is the number of decimal places. Since $\dfrac{a}{b}$ is reduced, we have that $b \mid 10^k$. To divide a power of 10 means that the only primes that can appear in its factorization are 2 and 5.

   ($\Leftarrow$) Write $\dfrac{a}{b} = \dfrac{a}{2^k 5^n}$. We can then multiply by the appropriate power of 2 or 5 in order to write this over a power of 10 and then write it as a terminating decimal. ■

Sets

1. $\mathbb{N} = \{1, 2, 3, ...\}$ (natural numbers)
2. $\mathbb{W} = \{0, 1, 2, 3, ...\}$ (whole numbers)
3. $\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$ (integers)
4. $\mathbb{Q} = \left\{\frac{a}{b} \mid a, b \in \mathbb{Z} \text{ and } b \neq 0\right\}$ (rational numbers) (or you can just say that these are the fractions)
5. $\mathbb{R}$ are the real numbers, the numbers that can be written as a decimal.

Properties

Let $*$ represent any of $+$, $-$, $\times$, or $\div$.

1. Closure - If you take two numbers a and b from a set, then $a * b$ is also in the set.
2. Commutative - For a and b in your set, $a * b = b * a$.
3. Associative - For a, b, and c in your set, $(a * b) * c = a * (b * c)$.
4. Identity - Let a be an element from your set. For addition, 0 is the identity, so $0 + a = a + 0 = a$. For multiplication, 1 is the identity, so $1 \times a = a \times 1 = a$. The identity doesn't exist for subtraction or division since $0 - a \neq a$ and $1 \div a \neq a$.
5. Inverse - Let a be an element from your set. For addition, $-a$ is the inverse, so $-a + a = a + -a = 0$. For multiplication, $\frac{1}{a}$ is the inverse, so $\frac{1}{a} \times a = a \times \frac{1}{a} = 1$. For subtraction and division, since we don't have an identity, we don't talk about the inverse.
6. Distributive - For a, b, and c in your set, $a \times (b + c) = a \times b + a \times c$.
7. Zero Product - For a in your set, $0 \times a = a \times 0 = 0$.
8. See example property questions.

Example Property Questions

1. True, since addition is always commutative.
2. True, since reciprocals of decimal numbers are decimal numbers.
3. False, $1 - 3 = -2 \notin \mathbb{W}$.
4. False, because $0 \notin \mathbb{N}$.
5. True, since the distributive property holds with pretty much all sets.
6. False, the only number that works is 1 since $a \div 1 = a$, but $1 \div a \neq a$.
7. True, as addition is always associative.
8. False, as 2 does not have its multiplicative inverse $\left(\frac{1}{2}\right)$ in $\mathbb{Z}$.
9. False, the only number that works is 0 since $a - 0 = a$, but $0 - a = -a$.
10. True, as the product of any two natural numbers is a natural number.
11. True, as $0 \in \mathbb{R}$.
12. False, as $0 - 1 \neq 1 - 0$.

Questions from the Classroom

1. Show her that the distributive property is specifically for *addition* inside the parentheses and does not work otherwise.
2. Using Base 10 blocks, explain that division by 10 is like putting these blocks in 10 groups. You can take each block and split it into 10 smaller pieces, so this makes each number lose a place value. Since there were no units, that disappears. However, this only works because there were no units, so it is important that the zero is at the end.
3. $b - a = -(a - b)$ and $a - b = -(b - a)$ (using the distributive property).
4. The negative sign stays with the b, so we actually would get $^{-}b + a$, which is not the same as $b - a$.
5. The student needs to carefully learn the order of operations, and multiplication comes before addition. (Side thought, order of operations is made up just to make things work well. We equally could have made them so that what was done was correct, but it would have been harder to do more complex math.)
6. Show Bob that 121 is not prime but is not divisible by any of those numbers. If he just wants to add 11, point out that you can keep using larger primes and make it not work with his set of things to check.
7. This is only true when the denominators are the same because they need to have the same number of pieces to compare them. A counter-example would be $\frac{3}{10} < \frac{1}{3}$.

8. Ken is thinking that the figure represents one whole, so draw another rectangle cut into 12 pieces and use that to distinguish between $\frac{11}{12} and \frac{11}{4}$.
9. This is an example that division doesn't always make things smaller. As a matter of fact, when we divide by a positive number that is less than 1, the number will always get bigger.
10. Write both of these numbers as fractions over 100 and show that when we compare numerators, $36 < 90$. Another way to think about it is to make sure the number of decimal places match, so compare 0.36 to 0.90.
11. The calculator can only perform to a certain amount of accuracy, but the fractions are not equal because the first is $1 - \frac{1}{9444}$ and the second is $1 - \frac{1}{9445}$.
12. We can raise the price as much as we want. A 100% increase would be doubling the price, but we can do more than double the price. A decrease, however, is taken from the actual amount. Taking 100% from a number would make it 0, so we can't take any more than that.
13. If the test has 20 questions, we can still write it as something out of 100 in order to write it as a percent by the Fundamental Law of Fractions.